Sobolev Space Reading Course Notes

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Preface

Herein I present my understanding of section 5.4 in 'Partial Differential Equations' by L.C. Evans [2]. The section proves a theorem for a constructing linear operator which extends functions in $W^{1,p}(U)$ to functions in $W^{1,p}(\mathbb{R}^n)$ where $U \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$. Throughout, I will point out points of interests and references which may offer more insight. I do not intend for this to be more than one person's attempt at understanding Evan's treatment. Please refer to the book to clear up details. You are welcome to email me about errors or for more discussion.

Extension Theorem

Suppose $1 \leq p \leq \infty$. Assume U is bounded and ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator,

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n) \tag{1}$$

such that for each $u \in W^{1,p}(U)$

i. $Eu = u$ *a.e.* in U

ii. Eu has support within V

iii. and

$$||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C||Eu||_{E:W^{1,p}(U)},$$

the constant C depending only on p, U, and V.

Thoughts on the Assumptions

The assumptions of this theorem are conditions which imply the existence of an extension. The goal is to extend functions in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$ with $\Omega \subset \mathbb{R}^n$. As Evans points out, extending the functions by allowing them to be zero outside of Ω causes functions to potentially become non-differentiable on along boundaries which prevents us from moving derivatives to test functions. I attempted to address why one would expect these conditions and if they can be weakened. After searching the internet, I found an example of which shows the need for these assumptions.

Unbounded and non-Lipschitz Boundary, uses Sobolev Imbedding theorem

Notes from a R. Hoppe [1] contain the following example of an unbounded function without a C^1 boundary which contradicts the Sobolev Imbedding theorem. We have not cover this theorem, but I think it's nice to have an example in hand. Consider the following function

$$u(x) = x_1^{-\epsilon/p}, \quad 0 < \epsilon < r$$

on a subset of $\mathbb{R}^2$:

$$\Omega = \{x = (x_1, x_2) | 0 < x_1 < 1, |x_2| < x_1^r, r > 1\}$$

Due to the cusp at the origin the domain is not C^1. The function, u, is in $W^{(1,p)}(\Omega)$.

$$D^{(1,0)}u = -\frac{\epsilon}{p} x_1^{-\frac{\epsilon}{p} - \frac{p}{p}}, \quad D^{(0,1)}u = 0$$

$$\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^p dx = -\frac{\epsilon}{p} \int_0^1 \int_{-x_1^r}^{x_1^r} (x_1^{-\frac{\epsilon}{p} - \frac{p}{p}})^p dx_2 dx_1 = -2\frac{\epsilon}{p} \int_0^1 x_1^{-\epsilon - p + r} dx_1$$

We see that, if $\epsilon < r + 1 - p$, then $||u|||_{W^{1,p}(\Omega)}$. But, $u \notin L^\infty(\Omega)$ with any extension, which according to the Sobolev Imbedding Theorem, implies that there cannot exist an extension to W.

It is also worth noting that the notes have an alternative extend which assumes Lipschitz Boundary conditions.

Thoughts on the Proof

Evans breaks up his proof into 7 steps. I will state steps and comment on them.

Step 1: Flatness Assumption and Local Decomposition

Fix $x^0 \in \partial U$ and suppose first, ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$. Then we may assume there exists an open ball B, with center x^0 and radius r such that

$$\begin{cases} B^+ := B \cap \{x_n \geq 0\} \subset \overline{U} \\ B^+ := B \cap \{x_n \leq 0\} \subset \mathbb{R} - U \end{cases}$$

Step 2: Local Extension and Boundary Assumption

Temporarily suppose that also $u \in C^\infty(\overline{U})$. We define then

$$\overline{u}(x) := \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, ..., x_{n-1}, -x_n) + 4u(x_1, ..., x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^- \end{cases}$$

This formula for $\overline{u}(x)$ is chosen to work within this setting. Different assumptions will require different choices of $\overline{u}(x)$.

Step 3: Imposing C^1 along the Boundary

We claim $\overline{u} \in C^1(B)$. Write $u^- := \overline{u}|_{B^-}$ and $u^+ := \overline{u}|_{B^+}$ and demonstrate first $u^-_{x_n} = u^+_{x_n}$ on $\{x_n = 0\}$.

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u}{\partial x_n}(x_1, ..., x_{n-1}, -x_n) - 2\frac{\partial u}{\partial x_n}(x_1, ..., -x_{n-1}, -x_n/2)$$

which shows $u^-_{x_n} = u^+_{x_n}$ on $x_n = 0$.

$$\frac{\partial u^-}{\partial x_i}(x) = -3\frac{\partial u}{\partial x_i}(x_1, ..., x_{n-1}, -x_n) + 4\frac{\partial u}{\partial x_i}(x_1, ..., -x_{n-1}, -x_n/2)$$

Setting $x_n = 0$, $u^-_{x_i} = u^+_{x_i}$ and $u^- = u^+$ when $x_n = 0$. Which implies that

$$D^\alpha u^-|_{x_n=0} = D^\alpha u^+|_{x_n=0}, \quad |\alpha| \leq 1$$

or $\overline{u} \in C^1(B)$.

My own version of the calculation. Be wary of mistakes

Step 4: Bounding the Extension

Goal:

$$||\overline{u}||_{W^{1,p}(B)} \leq C||u||_{W^{1,p}(B+)}$$

Where C does not depend on u.

Since $W^{1,p}(B)$ is Banach, we may write the following:

$$||\overline{u}||_{W^{1,p}(B)} = ||-3u(...,-x_n)+4u(...,-x_n/2)||_{W^{1,p}(B)} \leq 3||...||_{W^{1,p}(B)} + 4||...||_{W^{1,p}(B)}.$$

the first term of Eu is a reflection.

$$||u(...,-x_n)||^p_{W^{1,p}(B)} = \sum_{|\alpha|<1} \int_{B^-} |D^\alpha u(...,-x_n)|^p dx + \sum_{|\alpha|<1} \int_{B^+} |D^\alpha u|^p dx = 2||u||^p_{W^{1,p}(B^+)}$$

The second term requires a linear change of variables, ($y_1 = x_1, ..., y_n = -x_n/2$). This has a Jacobian of 2 and a differential operator changes from D^α to $2D^\alpha$. After the change of variables the integral is over half the lower ball. Note, integrating over the whole ball will give us a larger value. This is observation can be used for the upper bound. Take r to be the radius of the ball.

$$||u(...,-x_n/2)||^p_{W^{1,p}(B)} = \sum_{|\alpha|<1} \int_{B^-} |D^\alpha u(...,-x_n/2)|^p dx + \sum_{|\alpha|<1} \int_{B^+} |D^\alpha u(...,-x_n/2)|^p dx =$$

$$\sum_{|\alpha|<1} 2 \int_{B^- \cap \{x_n < r/2\}} |2^{-p} D^\alpha u|^p dy + \sum_{|\alpha|<1} \int_{B^+} |D^\alpha u|^p dx \leq (2^{1-p}+1) \int_{B^+} |D^\alpha u|^p dx$$

Might have made a mistake somewhere but none of these arguments needed any information on u.

Step 5: Relaxing the Flatness Assumption

Let us next consider the situation that ∂U is not necessarily flat near x^0. Then utilizing the notation and terminology from section $C.1$, we can find a C^1 mapping Φ, with inverse Ψ, such that Φ 'straightens out ∂U near x^0'.

$y = \Phi(x)$ and $u'(y) := u(\Psi(y))$. Apply the mapping first. Extend as before.

$$||\overline{u}'||_{W^{1,p}(B)} \leq C||u'||_{W^{1,p}(B^+)}$$

Convert back with $W = \Psi(B)$,

$$||\overline{u}||_{W^{1,p}(W)} \leq C||u||_{W^{1,p}(U)}$$

I also believe this is where the C^1 assumption is used. Note each of the B^+ sets should map inside of U when Ψ is applied. Note each constant depends on the selected set.

Step 6: Global Extension

Since ∂U is compact (closed and bounded) there exists finitely many points $x_i^0 \in \partial U$, open sets W_i, and extensions $\overline{u}_i$ of u to W_i.

Take $W_0 \subset\subset U$ so that $U \subset \cup_{i=0} n W_i$, and let $\{\xi\}_{i=0}^N$ be an associated partition of unity.

Write $u := \sum_{i=0}^N \xi_i \overline{u}_i$ with $\overline{u}_0 = u$.

Since there is a finite cover and the function is zero outside the support of the partitions of unity, we may write

$$||\overline{u}||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}$$

Note: I would like to add that one can generate the open cover on ∂U and apply the flattening argument in the follow way. First, take an open ball around each point in ∂U. Now, apply the flatting to each of them so that they are flatten at each point. In the flatten domain, each set is still open. Each open set contains a ball around the point to which the center of the original ball is mapped. Apply the bound from step 4 within each ball. Now, apply the inverse to each ball subset to find an open cover the original boundary. Now, we have an open cover of the original set. Apply the arguments from Step 6.

Step 7: Relaxing the Boundary Assumption by Approximation

Using Theorem 3 of Section 5.3.3, $u \in W^{1,p}(U)$ can be approximated by $u_m \in C^\infty(\overline{U})$ so that $u_m \to u$ in $W^{1,p}(U)$. By linearity of E and previous estimates,

$$||Eu_m - Eu_l||W^{1,p}(\mathbb{R}) = ||E(u_m - u_l)||W^{1,p}(\mathbb{R}) \leq C||u_m - u_l||W^{1,p}(U)$$

Thus Eu_m is a Cauchy sequence on $W^{1,p}(\mathbb{R})$ and converges to Eu.

I did not address the $p = \infty$ case.

Closing Remarks

Evan's comment on $k \geq 2$

If ∂U is C^2 then E will map $W^{(2,p)}(U)$ to $W^{(2,p)}(\mathbb{R}^n)$. Other assumptions are the same. However, the same construction cannot be applied to show that $W^{(k,p)}(U)$ can be extended to $W^{(k,p)}(\mathbb{R}^n)$ for $k > 2$. Higher order reflection methods can be applied.

References

[1] Notes on sobolev spaces. https://www.math.uh.edu/ rohop/Fall$_1$6$/downloads/Chapter2.pdf. Accessed:$ $2010-09-30$.

[2] Lawrence C Evans. Partial differential equations (graduate studies in mathematics, vol. 19). In *American Math. Soc*, 1998.